

Jensen's inequality for conditional expectations

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March 22 2005

Abstract

We study conditional expectations generated by an abelian C^* -subalgebra in the centralizer of a positive functional. We formulate and prove Jensen's inequality for functions of several variables with respect to this type of conditional expectations, and we obtain as a corollary Jensen's inequality for expectation values.

1 Preliminaries

An n -tuple $\underline{x} = (x_1, \dots, x_n)$ of elements in a C^* -algebra \mathcal{A} is said to be abelian if the elements x_1, \dots, x_n are mutually commuting. We say that an abelian n -tuple \underline{x} of self-adjoint elements is in the domain of a real continuous function f of n variables defined on a cube of real intervals $\underline{I} = I_1 \times \dots \times I_n$ if the spectrum $\sigma(x_i)$ of x_i is contained in I_i for each $i = 1, \dots, n$. In this situation $f(\underline{x})$ is naturally defined as an element in \mathcal{A} in the following way. We may assume that \mathcal{A} is realized as operators on a Hilbert space and let

$$x_i = \int \lambda dE_i(\lambda) \quad i = 1, \dots, n$$

denote the spectral resolutions of the operators x_1, \dots, x_n . Since the n -tuple $\underline{x} = (x_1, \dots, x_n)$ is abelian, the spectral measures E_1, \dots, E_n are mutually commuting. We may thus set

$$E(S_1 \times \dots \times S_n) = E_1(S_1) \dots E_n(S_n)$$

for Borel sets S_1, \dots, S_n in \mathbf{R} and extend E to a spectral measure on \mathbf{R}^n with support in \underline{I} . Setting

$$f(\underline{x}) = \int f(\lambda_1, \dots, \lambda_n) dE(\lambda_1, \dots, \lambda_n)$$

and since f is continuous, we finally realize that $f(\underline{x})$ is an element in \mathcal{A} .

2 Conditional expectations

Let \mathcal{C} be a separable abelian C^* -subalgebra of a C^* -algebra \mathcal{A} , and let φ be a positive functional on \mathcal{A} such that \mathcal{C} is contained in the centralizer

$$\mathcal{A}^\varphi = \{y \in \mathcal{A} \mid \varphi(xy) = \varphi(yx) \quad \forall x \in \mathcal{A}\}.$$

The subalgebra is of the form $\mathcal{C} = C_0(S)$ for some locally compact metric space S .

Theorem 2.1. *There exists a positive linear mapping*

$$(1) \quad \Phi: M(\mathcal{A}) \rightarrow L^\infty(S, \mu_\varphi)$$

on the multiplier algebra $M(\mathcal{A})$ such that

$$\Phi(xy) = \Phi(yx) = \Phi(x)y \quad x \in M(\mathcal{A}), y \in \mathcal{C}$$

almost everywhere, and a finite Radon measure μ_φ on S such that

$$(2) \quad \int_S z(s)\Phi(x)(s) d\mu_\varphi(s) = \varphi(zx) \quad z \in \mathcal{C}, x \in M(\mathcal{A}).$$

Proof. By the Riesz representation theorem there is a finite Radon measure μ_φ on S such that

$$\varphi(y) = \int_S y(s) d\mu_\varphi(s) \quad y \in \mathcal{C} = C_0(S).$$

For each positive element x in the multiplier algebra $M(\mathcal{A})$ we have

$$0 \leq \varphi(yx) = \varphi(y^{1/2}xy^{1/2}) \leq \|x\|\varphi(y) \quad y \in \mathcal{C}_+.$$

The functional $y \rightarrow \varphi(yx)$ on \mathcal{C} consequently defines a Radon measure on S which is dominated by a multiple of μ_φ , and it is therefore given by a unique element $\Phi(x)$ in $L^\infty(S, \mu_\varphi)$. By linearization this defines a positive linear mapping defined on the multiplier algebra

$$(3) \quad \Phi: M(\mathcal{A}) \rightarrow L^\infty(S, \mu_\varphi)$$

such that

$$\int_S z(s)\Phi(x)(s) d\mu_\varphi(s) = \varphi(zx) \quad z \in \mathcal{C}, x \in M(\mathcal{A}).$$

Furthermore, since

$$\int_S z(s)\Phi(yx)(s) d\mu_\varphi(s) = \varphi(zyx) = \int_S z(s)y(s)\Phi(x)(s) d\mu_\varphi(s)$$

for $x \in M(\mathcal{A})$ and $z, y \in \mathcal{C}$ we derive $\Phi(yx) = y\Phi(x) = \Phi(x)y$ almost everywhere. Since \mathcal{C} is contained in the centralizer \mathcal{A}^φ and thus $\varphi(zxy) = \varphi(yzx)$, we similarly obtain $\Phi(xy) = \Phi(x)y$ almost everywhere. **QED**

Note that $\Phi(z)(s) = z(s)$ almost everywhere in S for each $z \in \mathcal{C}$, cf. [6, 4, 5]. With a slight abuse of language we call Φ a conditional expectation even though its range is not a subalgebra of $M(\mathcal{A})$.

3 Jensen's inequality

Following the notation in [5] we consider a separable C^* -algebra \mathcal{A} of operators on a (separable) Hilbert space H , and a field $(a_t)_{t \in T}$ of operators in the multiplier algebra

$$M(\mathcal{A}) = \{a \in B(H) \mid a\mathcal{A} + \mathcal{A}a \subseteq \mathcal{A}\}$$

defined on a locally compact metric space T equipped with a Radon measure ν . We say that the field $(a_t)_{t \in T}$ is weak*-measurable if the function $t \rightarrow \varphi(a_t)$ is ν -measurable on T for each $\varphi \in \mathcal{A}^*$; and we say that the field is continuous if the function $t \rightarrow a_t$ is continuous [4].

As noted in [5] the field $(a_t)_{t \in T}$ is weak*-measurable, if and only if for each vector $\xi \in H$ the function $t \rightarrow a_t \xi$ is weakly (equivalently strongly) measurable. In particular, the composed field $(a_t^* b_t)_{t \in T}$ is weak*-measurable if both $(a_t)_{t \in T}$ and $(b_t)_{t \in T}$ are weak*-measurable fields.

If for a weak*-measurable field $(a_t)_{t \in T}$ the function $t \rightarrow |\varphi(a_t)|$ is integrable for every state $\varphi \in S(\mathcal{A})$ and the integrals

$$\int_T |\varphi(a_t)| d\nu(t) \leq K \quad \forall \varphi \in S(\mathcal{A})$$

are uniformly bounded by some constant K , then there is a unique element (a C^* -integral in Pedersen's terminology [8, 2.5.15]) in the multiplier algebra $M(\mathcal{A})$, designated by

$$\int_T a_t d\nu(t),$$

such that

$$\varphi \left(\int_T a_t d\nu(t) \right) = \int_T \varphi(a_t) d\nu(t) \quad \forall \varphi \in \mathcal{A}^*.$$

We say in this case that the field $(a_t)_{t \in T}$ is integrable. Finally we say that a field $(a_t)_{t \in T}$ is a unital column field [1, 4, 5], if it is weak*-measurable and

$$\int_T a_t^* a_t d\nu(t) = 1.$$

We note that a C^* -subalgebra of a separable C^* -algebra is automatically separable.

Theorem 3.1. *Let \mathcal{C} be an abelian C^* -subalgebra of a separable C^* -algebra \mathcal{A} , φ be a positive functional on \mathcal{A} such that \mathcal{C} is contained in the centralizer \mathcal{A}^φ and let*

$$\Phi: M(\mathcal{A}) \rightarrow L^\infty(S, \mu_\varphi)$$

be the conditional expectation defined in (3). Let furthermore $f: \underline{I} \rightarrow \mathbf{R}$ be a continuous convex function of n variables defined on a cube, and let $t \rightarrow a_t \in M(\mathcal{A})$ be a unital column field on a locally compact Hausdorff space T with a Radon measure ν . If $t \rightarrow \underline{x}_t$ is an essentially bounded, weak measurable field on T of abelian n -tuples of self-adjoint elements in \mathcal{A} in the domain of f , then*

$$(4) \quad f(\Phi(y_1), \dots, \Phi(y_n)) \leq \Phi \left(\int_T a_t^* f(\underline{x}_t) a_t d\nu(t) \right)$$

almost everywhere, where the n -tuple \underline{y} in $M(\mathcal{A})$ is defined by setting

$$\underline{y} = (y_1, \dots, y_n) = \int_T a_t^* \underline{x}_t a_t d\nu(t).$$

Proof. The subalgebra \mathcal{C} is as noted above of the form $\mathcal{C} = C_0(S)$ for some locally compact metric space S , and since the C^* -algebra $C_0(\underline{I})$ is separable we may for almost every s in S define a Radon measure μ_s on \underline{I} by setting

$$\mu_s(g) = \int_{\underline{I}} g(\underline{\lambda}) d\mu_s(\underline{\lambda}) = \Phi \left(\int_T a_t^* g(\underline{x}_t) a_t d\mu(t) \right) (s) \quad g \in C_0(\underline{I}).$$

Since

$$\mu_s(1) = \Phi \left(\int_T a_t^* a_t d\mu(t) \right) = \Phi(1) = 1$$

we observe that μ_s is a probability measure. If we put $g_i(\underline{\lambda}) = \lambda_i$ then

$$\int_{\underline{I}} g_i(\underline{\lambda}) d\mu_s(\underline{\lambda}) = \Phi \left(\int_T a_t^* x_{it} a_t d\mu(t) \right) (s) = \Phi(y_i)(s)$$

for $i = 1, \dots, n$ and since f is convex we obtain

$$\begin{aligned}
f(\Phi(y_1)(s), \dots, \Phi(y_n)(s)) &= f\left(\int_{\underline{I}} g_1(\underline{\lambda}) d\mu_s(\underline{\lambda}), \dots, \int_{\underline{I}} g_n(\underline{\lambda}) d\mu_s(\underline{\lambda})\right) \\
&\leq \int_{\underline{I}} f(g_1(\underline{\lambda}), \dots, g_n(\underline{\lambda})) d\mu_s(\underline{\lambda}) = \int_{\underline{I}} f(\underline{\lambda}) d\mu_s(\underline{\lambda}) \\
&= \Phi\left(\int_T a_t^* f(\underline{x}_t) a_t d\mu(t)\right)(s)
\end{aligned}$$

for almost all s in S .

QED

The following corollary is known as "Jensen's inequality for expectation values". It was formulated (for continuous fields) in the reference [3], where a more direct proof is given.

Corollary 3.2. *Let $f : \underline{I} \rightarrow \mathbf{R}$ be a continuous convex function of n variables defined on a cube, and let $t \rightarrow a_t \in B(H)$ be a unital column field on a locally compact Hausdorff space T with a Radon measure ν . If $t \rightarrow \underline{x}_t$ is a bounded weak*-measurable field on T of abelian n -tuples of self-adjoint operators on H in the domain of f , then*

$$(5) \quad f((y_1\xi \mid \xi), \dots, (y_n\xi \mid \xi)) \leq \left(\int_T a_t^* f(\underline{x}_t) a_t d\nu(t) \xi \mid \xi \right)$$

for any unit vector $\xi \in H$, where the n -tuple \underline{y} is defined by setting

$$\underline{y} = (y_1, \dots, y_n) = \int_T a_t^* \underline{x}_t a_t d\nu(t).$$

Proof. The statement follows from Theorem 3.1 by choosing φ as the trace and letting \mathcal{C} be the C^* -algebra generated by the orthogonal projection P on the vector ξ . Then $\mathcal{C} = C_0(S)$ where $S = \{0, 1\}$, and an element $z \in \mathcal{C}$ has the representation

$$z = z(0)P + z(1)(1 - P).$$

The measure $d\mu_\varphi$ gives unit weight in each of the two points, and the conditional expectation Φ is given by

$$\Phi(x)(s) = \begin{cases} (x\xi \mid \xi) & s = 0 \\ \text{Tr}(x - Px) & s = 1. \end{cases}$$

Indeed,

$$\begin{aligned}
\varphi(zx) &= \text{Tr}\left((z(0)P + z(1)(1 - P))x\right) \\
&= z(0)\Phi(x)(0) + z(1)\Phi(x)(0) \\
&= \int_S z(s)\Phi(x)(s) ds
\end{aligned}$$

as required. The statement follows by evaluating the functions appearing on each side of the inequality (4) in the point $s = 0$. **QED**

Remark 3.3. *If we choose ν as a probability measure on T , then the trivial field $a_t = 1$ for $t \in T$ is unital and (5) takes the form*

$$f\left(\left(\int_T x_{1t} d\nu(t)\xi \mid \xi\right), \dots, \left(\int_T x_{nt} d\nu(t)\xi \mid \xi\right)\right) \leq \left(\int_T f(\underline{x}_t) d\nu(t)\xi \mid \xi\right)$$

for bounded weak*-measurable fields of abelian n -tuples $\underline{x}_t = (x_{1t}, \dots, x_{nt})$ of self-adjoint operators in the domain of f and unit vectors ξ . By choosing ν as an atomic measure with one atom we get a version

$$(6) \quad f((x_1\xi \mid \xi), \dots, (x_n\xi \mid \xi)) \leq (f(\underline{x})\xi \mid \xi)$$

of the Jensen inequality by Mond and Pečarić [7]. By further considering a direct sum

$$\xi = \bigoplus_{j=1}^m \xi_j \quad \text{and} \quad x = (x_1, \dots, x_n) = \bigoplus_{j=1}^m (x_{1j}, \dots, x_{nj})$$

we obtain the familiar version

$$f\left(\sum_{j=1}^m (x_{1j}\xi_j \mid \xi_j), \dots, \sum_{j=1}^m (x_{nj}\xi_j \mid \xi_j)\right) \leq \sum_{j=1}^m (f(x_{1j}, \dots, x_{nj})\xi_j \mid \xi_j)$$

valid for abelian n -tuples (x_{1j}, \dots, x_{nj}) , $j = 1, \dots, m$ of self-adjoint operators in the domain of f and vectors ξ_1, \dots, ξ_m with $\|\xi_1\|^2 + \dots + \|\xi_m\|^2 = 1$.

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